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EQUATIONS OF GASDYNAMICS IN A NONINERTIAL DEFORMABLE COORDINATE SYSTEM

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Generalized equations of gasdynamics are obtained in differential form in a relative curvilinear deformable coordinate system and, also, in divergent form in a system of coordinates rotating at constant speed. It is shown that the known forms of gasdynamic equations are particular cases of generalized equations.

Two approaches are discernible in the determination of unsteady multidimensional gasdynamic flows in regions of complex geometry. In one of these the physical region is subdivided into separate subregions in each of which the flow is smooth and defined by a system of differential equations, and at whose boundaries relationships at shock waves and contact discontinuities are satisfied. In the second approach continuous calculation is used, in which equations of gasdynamics are of the form of conservation laws throughout the calculation region.

In the first method the calculation accuracy can be improved by separating flow singularities (shock waves, contact discontinuities) relating their position to a system of curvilinear coordinates. The problem of further extension of the equations of gasdynamics presented in [1, 2] to the use of arbitrary deformable coordinate system arises in the case of unsteady flows.

In the continuous calculation method the equations of gasdynamics are of the form of conservation laws defined in Cartesian [1], orthogonal curvilinear [1, 3], or arbitrary curvilinear [2, 4] coordinate systems. An important particular case (e. g., in the theory of bladed machines) is that of the noninertial coordinate system rotating at constant angular velocity.

1. The arbitrary deformable coordinate system. Along with the Cartesian system of coordinates x^i with basis vectors $\mathbf{x}_i = \mathbf{x}^i$ we shall use a curvilinear deformable system of coordinates $q^i(x, t)$ with basis vectors $\mathbf{e}_i$ and $\mathbf{e}^i$ (the super- and subscripts relate to contravariant and covariant basis vectors [2, 5], respectively)

$$\mathbf{e}^i = a_\beta{}^i \mathbf{x}^\beta, \quad \mathbf{e}_i = b_i{}^\beta \mathbf{x}_\beta \tag{1.1}$$

$$a_\beta{}^i = \frac{\partial q^i}{\partial x^\beta}, \quad b_i{}^\beta = \frac{\partial x^\beta}{\partial q^i}, \quad a_\beta{}^i b_i{}^\beta = \delta_j{}^i \tag{1.2}$$

where $a_\beta^{\,i}$ and $b_i^{\,\beta}$ are matrices of direct and inverse transformation of coordinates, and summation is carried out over recurrent indices.

The system of equations is formulated in terms of contravariant components of the absolute $\mathbf{V} = V^i \mathbf{e}_i$, relative $\mathbf{V}_r = V_r^{\,i} \mathbf{e}_i$ and carrier (*) $\mathbf{V}_e = V_e^{\,i} \mathbf{e}_i$ velocity vectors.

In the system of coordinates q^i the law of mass conservation in time t for the fluid volume τ is of the form

$$\frac{d}{dt}\int\limits_\tau \rho \sqrt{g}\, dq^1\, dq^2\, dq^3 = 0, \quad g = \det \| g_{ij} \| \tag{1.3}$$

where ρ is the medium density and g is the determinant of the metric tensor [2, 5].

The product of coordinate differentials $dq^1 dq^2 dq^3$ in Eq. (1.3) may be considered as some elementary "volume" in the system of coordinates q^i which is then assumed to be Cartesian (of course unrelated to the x^i -system). We apply to the left-hand side of (1.3) the rule of differentiation of an integral with respect to a moving volume [2] and obtain

$$\frac{d}{dt}\int\limits_\tau \rho \sqrt{g}\, dq^1\, dq^2\, dq^3 =$$

$$\int\limits_\tau \left[\frac{\partial (\rho \sqrt{g})}{\partial t} + \operatorname{div} (\rho \sqrt{g} \mathbf{V}_r) \right] dq^1\, dq^2\, dq^3 = 0$$

where $\partial / \partial t$ is a partial derivative with respect to time in the related coordinate system; the operation div is determined, as in Cartesian coordinates, by

$$\operatorname{div} (\rho \sqrt{g} \mathbf{V}_r) = \partial (\rho \sqrt{g} V_r^{\,i}) / \partial q^i.$$

Since the volume can be arbitrary, we write the equation of mass conservation as

$$\frac{\partial (\rho \sqrt{g})}{\partial t} + \frac{\partial (\rho \sqrt{g} V_r^{\,i})}{\partial q^i} = 0 \tag{1.4}$$

To have the equation of motion

$$\mathbf{a} = -\frac{1}{\rho} \operatorname{grad} p \tag{1.5}$$

in hydrodynamical form it is necessary to define the absolute acceleration $\mathbf{a} = d\,(V^i \mathbf{e}_i) / dt$ in the system of coordinates q^i. When taking the total derivative with respect to time it is necessary to take into account that the basis vectors of a fluid

*) Translator's note: In turbomachinery usually called the peripheral velocity.

particle are represented in the form

$$V^i = V^i\,[q\,(t),\,t],\quad \mathbf{e}_i = \mathbf{e}_i\,[q\,(t),\,t]$$

where $q^\alpha = q^\alpha\,(t)$ is the equation of the trajectory of a specified fluid particle, i.e. $\partial q^\alpha\,/\,dt = V_r^{\,\alpha}$. Hence

$$\mathbf{a} = \frac{d\,(V^i\mathbf{e}_i)}{dt} = \frac{dV^i}{dt}\,\mathbf{e}_i + V^i\,\frac{d\mathbf{e}_i}{dt} = \tag{1.6}$$

$$\left(\frac{\partial V^i}{\partial t} + \frac{\partial V^i}{\partial q^\alpha}\frac{dq^\alpha}{dt}\right)\mathbf{e}_i + V^\alpha\left(\frac{\partial\mathbf{e}_\alpha}{\partial t} + \frac{\partial\mathbf{e}_\alpha}{\partial q^\beta}\frac{dq^\beta}{dt}\right) =$$

$$\left[\frac{\partial V^i}{\partial t} + V_r^{\,\alpha}\left(\frac{\partial V^i}{\partial q^\alpha} + V^\beta\Gamma^i_{\alpha\beta}\right)\right]\mathbf{e}_i + V^\alpha\,\frac{\partial\mathbf{e}_\alpha}{\partial t}$$

where $\Gamma^i_{\alpha\beta}$ are components of vector $\partial\mathbf{e}_\alpha\,/\,\partial q^\beta$ in basis $\mathbf{e}_i$ (Christoffel symbols).

To determine components of vector $\partial\mathbf{e}_\alpha\,/\,\partial t = c_\alpha^{\,j}\mathbf{e}_j$ we differentiate the relationship $\mathbf{e}^i\cdot\mathbf{e}_\alpha = \delta_\alpha^{\,i}$ taking into account (1.1), i.e.

$$\mathbf{e}^i\cdot c_\alpha^{\,j}\mathbf{e}_j + b_\alpha^{\,p}\mathbf{x}_p\cdot\,\frac{\partial a_\beta^{\,i}}{\partial t}\,\mathbf{x}^\beta = 0$$

From this

$$c_\alpha^{\,i} = -\,\frac{\partial a_\beta^{\,i}}{\partial t}\,b_\alpha^{\,\beta} \tag{1.7}$$

The substitution of (1.6) into (1.5) and the representation of the absolute velocity V^i as the sum of the relative $V_r^{\,i}$ and the carrier $V_e^{\,i}$ velocities yields in the noninertial deformable system of coordinates the following equation of motion:

$$\left\{\left[\frac{\partial V_r^{\,i}}{\partial t} + V_r^{\,\alpha}\left(\frac{\partial V_r^{\,i}}{\partial q^\alpha} + V_r^{\,\beta}\Gamma^i_{\alpha\beta}\right)\right] + \left[\frac{\partial V_e^{\,i}}{\partial t} + V_r^{\,\alpha}\left(\frac{\partial V_e^{\,i}}{\partial q^\alpha} + \right.\right.\right. \tag{1.8}$$

$$\left.\left.\left. V_e^{\,\beta}\Gamma^i_{\alpha\beta}\right)\right] + \left[(V_r^{\,\alpha} + V_e^{\,\alpha})\,c_\alpha^{\,i}\right]\right\}\mathbf{e}_i = -\,\frac{1}{\rho}\,\mathrm{grad}\,p$$

where $c_\alpha^{\,i}$ is determined by formulas (1.7).

To determine the contravariant components of the carrier velocity vector $V_e^{\,i}$ in the system of coordinates q^i we differentiate the identity $q^i = q^i\,[x\,(q,\,t),\,t]$ with respect to t. We have

$$\frac{\partial q^i}{\partial t} = -\,\frac{\partial q^i}{\partial x^j}\,\frac{\partial x^j}{\partial t}$$

Since $\partial x^j\,(q,\,t)\,/\,\partial t$ is the contravariant component of the carrier velocity vector in the system of coordinates x^i, hence, owing to the transformation of vector components when passing to another coordinate system, the quantity $\partial q^i(x, t) / \partial t$ taken with the minus sign is the covariant component of the carrier velocity vector in the system of coordinates q^i, i.e.

$$V_e^i = -\partial q^i(x, t) / \partial t$$

In the particular case of $\mathbf{V}_e = \text{const}$ and $c_\alpha^i = 0$ from (1.8) we obtain the known form of equations of motion in the inertial undeformable coordinate system

$$\left(\frac{\partial V^i}{\partial t} + V^\alpha \frac{\partial V^i}{\partial q^\alpha} + V^\alpha V^\beta \Gamma_{\alpha\beta}^i\right) \mathbf{e}_i = -\frac{1}{\rho} \operatorname{grad} p$$

Let us consider an undeformable coordinate system rotating at angular velocity $\boldsymbol{\omega}$. In that case the expression in the first set of brackets in (1.8) represents the total time derivative of the relative velocity in the respective coordinate system, i.e. $(dV_r^i \mathbf{e}_i / dt)_r$ is the relative acceleration. The expression in the second set of brackets represents the total time derivative of the carrier velocity

$$\left[\frac{\partial V_e^i}{\partial t} + V_r^\alpha \left(\frac{\partial V_e^i}{\partial q^\alpha} + V_e^\beta \Gamma_{\alpha\beta}^i\right)\right] \mathbf{e}_i = \left(\frac{dV_e^i \mathbf{e}_i}{dt}\right)_r = \tag{1.9}$$

$$\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_r + \left(\frac{d\boldsymbol{\omega}}{dt}\right)_r \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{V}_r + \boldsymbol{\varepsilon} \times \mathbf{r} \quad (\mathbf{V}_e = \boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{r}$ is the radius vector and $\boldsymbol{\varepsilon}$ is the angular acceleration.

We transform the last term in the left-hand side of (1.8)

$$(V_r^\alpha + V_e^\alpha) c_\alpha^i \mathbf{e}_i = (V_r^\alpha + V_e^\alpha) \frac{\partial \mathbf{e}_\alpha}{\partial t} = \tag{1.10}$$

$$(V_r^\alpha + V_e^\alpha)(\boldsymbol{\omega} \times \mathbf{e}_\alpha) = \boldsymbol{\omega} \times \mathbf{V}_r + \boldsymbol{\omega} \times \mathbf{V}_e$$

and, substituting (1.9) and (1.10) into (1.8), we obtain

$$\frac{d\mathbf{V}_r}{dt} + 2\boldsymbol{\omega} \times \mathbf{V}_r + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} + \boldsymbol{\varepsilon} \times \mathbf{r} = -\frac{1}{\rho} \operatorname{grad} p \tag{1.11}$$

When $\boldsymbol{\varepsilon} = 0$ we have the equation of motion in the system of coordinates rotating at constant angular velocity [6].

For a non-heat-conducting gas the law of energy conservation is equivalent to the conservation of entropy by a fluid particle in regions of smooth flow into which the whole considered region is divided by the introduced relative coordinate system

$$\frac{\partial S}{\partial t} + V_r^i \frac{\partial S}{\partial q^i} = 0 \tag{1.12}$$

The system of equations of continuity (1.4), motion (1.8), and conservation of energy (1.12) is closed by the equation of state.

2. **Divergent equations of gasdynamics in a curvilinear coordinate system rotating at constant velocity.** The divergent form of the equation of mass conservation in an arbitrary coordinate system is of the form (1.4).

The law of momentum conservation in time t for the fluid volume τ may be represented in the integral form

$$\frac{d}{dt}\int_\tau \rho \mathbf{V}\, d\tau = -\int_s p\mathbf{n}\, ds \equiv -\int_\tau \operatorname{grad} p\, d\tau \tag{2.1}$$

where $\mathbf{n}$ is the unit vector of the external normal to the s-surface that bounds the fluid volume τ.

We denote by $\mathbf{a}$, $\mathbf{a}_e$ and $\mathbf{a}_r = d\mathbf{V}_r / dt$ the absolute, carrier, and relative accelerations, respectively, and, taking into account (1.3), transform the first integral in (2.1) as follows:

$$\frac{d}{dt}\int_\tau \rho \mathbf{V}\, d\tau = \int_\tau \frac{d\mathbf{V}}{dt}\rho\, d\tau = \int_\tau \rho\,(\mathbf{a}_e + \mathbf{a}_r + 2\boldsymbol{\omega}\times \mathbf{V}_r)\, d\tau \tag{2.2}$$

$$\int_\tau \rho\mathbf{a}_r\, d\tau = \int_\tau \rho \frac{d\mathbf{V}_r}{dt}\, d\tau = \frac{d}{dt}\int_\tau \rho \mathbf{V}_r\, d\tau = \int_\tau \frac{\partial}{\partial t}(\rho\mathbf{V}_r)\, d\tau +$$

$$\int_s \rho\mathbf{V}_r(\mathbf{V}_r\mathbf{n})\, ds = \int_\tau \left[\frac{\partial(\rho\mathbf{V}_r)}{\partial t} + \operatorname{div}(\rho\mathbf{V}_r\mathbf{V}_r)\right] d\tau$$

where $\rho\mathbf{V}_r\mathbf{V}_r$ is a tensor dyad.

The equation of momentum conservation is obtained from (2.1) with allowance for (2.2) in the vector form

$$\partial / \partial t\, (\rho\mathbf{V}_r) + \operatorname{div}\,(\rho\mathbf{V}_r\mathbf{V}_r) + \operatorname{grad} p = -\rho\mathbf{a}_e - 2\rho\boldsymbol{\omega}\times \mathbf{V}_r \tag{2.3}$$

The law of energy conservation for the fluid volume τ in integral form is

$$\frac{d}{dt}\int_\tau \left(\rho\varepsilon + \rho\frac{V^2}{2}\right) d\tau = -\int_\tau p\,(\mathbf{V}\mathbf{n})\, ds \tag{2.4}$$

where ε is the internal energy.

Taking into account (1.5) we transform the integrals in (2.4) as follows:

$$\frac{d}{dt}\int_\tau \rho \frac{V^2}{2} d\tau = \int_\tau \rho \frac{d(\mathbf{V}\mathbf{V})}{2dt} d\tau = \int_\tau \rho (\mathbf{V}\mathbf{a})\, d\tau =$$
$$\int_\tau \rho \left[(\mathbf{V}_r + \mathbf{V}_e)(\mathbf{a}_e + \mathbf{a}_r + 2\boldsymbol{\omega} \times \mathbf{V}_r)\right] d\tau =$$
$$\int_\tau \rho \left[\mathbf{V}_r\mathbf{a}_e + \mathbf{V}_r\mathbf{a}_r + \mathbf{V}_e\mathbf{a} + 2\mathbf{V}_r(\boldsymbol{\omega} \times \mathbf{V}_r)\right] d\tau$$
$$\mathbf{V}_r(\boldsymbol{\omega} \times \mathbf{V}_r) \equiv 0, \quad \mathbf{V}_r\mathbf{a}_r = \frac{1}{2}\frac{d\mathbf{V}_r^2}{dt}, \quad \mathbf{V}_e\mathbf{a} = -\mathbf{V}_e \cdot \frac{1}{\rho}\,\mathrm{grad}\, p \, .$$

For the system of coordinates q^i rotating at constant angular velocity $\boldsymbol{\omega}$ (the metric g_{ij} is independent of time) we have

$$\mathbf{a}_e = -\mathrm{grad}\, u^2 / 2$$

where u is the linear velocity of points of the q^i-coordinate system. From (1.4) we then have

$$\partial\rho / \partial t + \mathrm{div}\,(\rho\mathbf{V}_r) = 0$$

which yields

$$\rho\mathbf{V}_r\mathbf{a}_e = -\rho\mathbf{V}_r\,\mathrm{grad}\,\frac{u^2}{2} = -\,\mathrm{div}\left(\rho\mathbf{V}_r\frac{u^2}{2}\right) +$$
$$\frac{u^2}{2}\,\mathrm{div}\,(\rho\mathbf{V}_r) = -\,\mathrm{div}\left(\rho\mathbf{V}_r\frac{u^2}{2}\right) - \frac{u^2}{2}\left(\frac{\partial\rho}{\partial t}\right)_{q^i} =$$
$$-\,\mathrm{div}\left(\rho\mathbf{V}_r\frac{u^2}{2}\right) - \left(\frac{\partial}{\partial t}\rho\frac{u^2}{2}\right)_{q^i}$$
$$\int_s p(\mathbf{V}\mathbf{n})\, ds = \int_\tau \mathrm{div}\,(p\mathbf{V})\, d\tau = \int_\tau \left[\mathrm{div}\,(p\mathbf{V}_r) + \mathrm{div}\,(\rho\mathbf{V}_e)\right] d\tau =$$
$$\int_\tau \left[\mathrm{div}\,(p\mathbf{V}_r) + p\,\mathrm{div}\,\mathbf{V}_e + \mathbf{V}_e\,\mathrm{grad}\, p\right] d\tau$$
$$\mathrm{div}\,\mathbf{V}_e = \mathrm{div}\,(\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{r}\cdot\mathrm{rot}\,\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathrm{rot}\,\mathbf{r}) \equiv 0$$
$$\int_\tau \rho\mathbf{V}_r\mathbf{a}_r d\tau = \int_\tau \rho\frac{1}{2}\left(\frac{dV_r^2}{dt}\right)_{q^i} d\tau = \frac{d}{dt}\left(\int_\tau \rho\frac{V_r^2}{2} d\tau\right)_{q^i} =$$
$$\int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\rho V_r^4)_{q^i} d\tau + \int_s \rho\frac{V_r^2}{2}(\mathbf{V}_r\mathbf{n})\, ds =$$
$$\int_\tau \left[\left(\frac{\partial}{\partial t}\rho\frac{V_r^2}{2}\right)_{q^i} + \mathrm{div}\left(\rho\mathbf{V}_r\frac{V_r^2}{2}\right)\right] d\tau$$

$$\frac{d}{dt}\int_\tau \rho\varepsilon\, d\tau = \int_\tau \left(\frac{\partial\rho\varepsilon}{\partial t}\right)_{q^i} + \int_s \rho\varepsilon\,(\mathbf{V}_r\mathbf{n})\, ds =$$
$$\int_\tau \left[\left(\frac{\partial\rho\varepsilon}{\partial t}\right)_{q^i} + \operatorname{div}(\rho\varepsilon\mathbf{V}_r)\right] d\tau$$

Substituting obtained expressions into (2.4) and taking into account the arbitrariness of the volume of integration τ, we obtain the equation of energy of the divergent form

$$\frac{\partial E}{\partial t} + \operatorname{div}[(E+p)\mathbf{V}_r] = 0, \quad E = \rho\left(\varepsilon + \frac{V_r^2 - u^2}{2}\right) \tag{2.5}$$

In the case of three space coordinates the symbolic vector form of divergent equations of unstable gasdynamics in an arbitrary noninertial coordinate system rotating at constant velocity is

$$\partial f / \partial t + \nabla_1 F_1 + \nabla_2 F_2 + \nabla_3 F_3 + H = 0 \tag{2.6}$$

$$f = \begin{Vmatrix} \rho \\ \rho V^1 \\ \rho V^2 \\ \rho V^3 \\ E \end{Vmatrix}, \quad F_1 = \begin{Vmatrix} \rho V^1 \\ \rho V^1 V^1 + g^{11} p \\ \rho V^1 V^2 + g^{12} p \\ \rho V^1 V^3 + g^{13} p \\ (E+p) V^1 \end{Vmatrix}$$

$$F_2 = \begin{Vmatrix} \rho V^2 \\ \rho V^2 V^1 + g^{21} p \\ \rho V^2 V^2 + g^{22} p \\ \rho V^2 V^3 + g^{23} p \\ (E+p) V^2 \end{Vmatrix}, \quad F_3 = \begin{Vmatrix} \rho V^3 \\ \rho V^3 V^1 + g^{31} p \\ \rho V^3 V^2 + g^{32} p \\ \rho V^3 V^3 + g^{33} p \\ (E+p) V^3 \end{Vmatrix}$$

$$H = \begin{Vmatrix} 0 \\ \rho a_e^1 + \frac{2}{\sqrt{g}}\rho(\omega_2 V^3 - \omega_3 V^2) \\ \rho a_e^2 + \frac{2}{\sqrt{g}}\rho(\omega_3 V^1 - \omega_1 V^3) \\ \rho a_e^3 + \frac{2}{\sqrt{g}}\rho(\omega_1 V^2 - \omega_2 V^1) \\ 0 \end{Vmatrix}$$

In an orthogonal system of coordinates we have $g_{ii} = H_i^2$ (no summation with respect to i), $g_{ik} = 0$, and $g = H_1^2 H_2^2 H_3^2$ (H_1, H_2, and H_3 are Lamé coefficients). As the result of transformations, system (2.6) can be represented in the form of equations of conservation of mass, momentum, and energy

$$\frac{\partial(H_1H_2H_3\rho)}{\partial t} + \frac{\partial}{\partial x^j}\left(\frac{H_1H_2H_3}{H_j}\,\rho V_j\right) = 0 \tag{2.7}$$
$$\frac{\partial}{\partial t}(H_1H_2H_3\rho V_i) + \frac{\partial}{\partial x^j}\left[\frac{H_1H_2H_3}{H_j}(\delta_{ij}p + \rho V_i V_j)\right] =$$

$$\frac{H_1H_2H_3}{H_iH_j}\frac{\partial H_j}{\partial x^i}\rho V_jV_j + p\frac{\partial}{\partial x^i}\left(\frac{H_1H_2H_3}{H_i}\right) -$$
$$\frac{H_1H_2H_3}{H_iH_j}\frac{\partial H_i}{\partial x^j}\rho V_iV_j - H_iH_1H_2H_3\rho\left(\frac{a_{ei}}{H_i} + 2e^{ijk}H_jH_k\omega_jV_k\right)$$
$$\frac{\partial(H_1H_2H_3E)}{\partial t} + \frac{\partial}{\partial x^j}\left[\frac{H_1H_2H_3}{H_j}(E+p)V_j\right] = 0$$
$$(i, j, k = 1, 2, 3)$$

where e^{ijk} is the Levi-Civita tensor.

The system of Eqs. (2.7) is closed by the equation of state.

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